Itô Calculus, application to Black-Sholes formula

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Outline

Gaussian White Noise Process

$$\frac{dx(t)}{dt} = \underbrace{A(x(t),t)}_{\text{Drift}} + \underbrace{\sqrt{D(x(t),t)}}_{\text{Diffusion}} \underbrace{\Gamma(t)}_{\text{White Noise Fct}} \quad (1)$$

With $\langle \Gamma(t) \rangle = 0$ and $\langle \Gamma(t)\Gamma(t') \rangle = \delta(t - t')$
Rewrite:

$$dx(t) = A(x(t),t)dt + \sqrt{D(x(t),t)}dW(t) \quad (2)$$

Where $dW(t) \equiv \Gamma(t)dt$ is a Wiener increment

Properties of a Wiener process $W(t)$

We write $W_t \equiv W(t)$

1. $W_0 = 0$
2. Independent increments: $\forall t > s \geq 0$, $W_t - W_s$ is independent of $\{W_u\}_{u \leq s}$
3. Normally distributed increments: $\forall t > s \geq 0$, $W_t - W_s \sim \mathcal{N}(0, t - s)$
4. W_t is continuous in t

Itô's Integral

Definition: on blackboard!

Main Properties:

1. Existence whenever G(t) is continuous and non-anticipating on $[0, t]$
2. General Differentiation Rule: for an arbitrary function $f(W(t), t)$:

$$df(W(t), t) = \left(\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial W^2} \right) dt + \frac{\partial f}{\partial W} dW \tag{3}$$

Why is $dW^2 = dt$? And why is $dW^{N+2} = 0$ for $N > 0$?
$\Rightarrow$ Proof and example on blackboard!

Itô's Lemma

Let's go back to the initial SDE:

$$dx(t) = A(x(t), t)dt + \sqrt{D(x(t), t)}dW(t) \tag{4}$$

For an arbitrary function $f(x(t))$:

$$df(x(t)) = f(x(t) + dx(t)) - f(x(t)) = \sum_{n=1}^{+\infty} \frac{1}{n!} \frac{\partial^n f(x(t))}{\partial x^n} (dx(t))^n \tag{5}$$

Intermediate steps on blackboard!

Itô's Lemma:

$$df(x(t)) = \left[A(x(t), t) f'(x(t)) + \frac{1}{2} D(x(t), t) f''(x(t)) \right] dt + \sqrt{D(x(t), t)} f'(x(t)) dW(t)$$

Outline

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Black-Scholes Model

- Black and Scholes (both economist and mathematician) published in 1973 the article *The Pricing of Options and Corporate Liabilities*[1] to introduce their model.
- M. Scholes got the Nobel Prize in 1997 for this work, along with Robert Merton (F. Black died in 1995).

Figure 1: Fischer Black and Myron Scholes

[1] Fischer Black and Myron Scholes. "The Pricing of Options and Corporate Liabilities". In: *Journal of Political Economy* 81.3 (1973), pp. 637–654.

European Option : definition

Definition A *European Option* is a contract that gives its owner the right[2] to buy or sell a certain asset at a predetermined delivery price (the *strike*, k) in a future time (the maturity date, T).
The goal of Black and Scholes was to give a value for such contract at time $t < T$.

[2] but not the obligation

European Option : value at time T

Let's consider a *call* option i.e. the option to buy the underlying.

- At maturity time T, the underlying is worth S_T on the market and the owner has the opportunity to buy it at k
  - If $S_T > k$ the owner should exercise the option, he will gain $S_T - k$ if he sells directly the asset.
  - On the contrary, if $S_T < k$ the owner has no interest to exercise the option : his payoff is 0.
- The payoff (gains at time T) is thus mathematically defined as $\Phi(S_T) \equiv (S_T - k)^+ = \max(S_T - k, 0)$:

European Call Option : Payoff

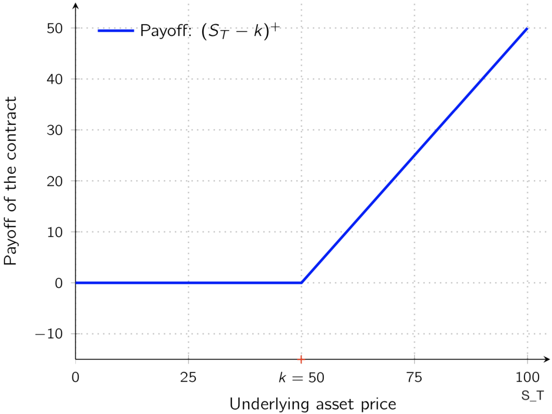

Figure 2: Payoff of a Long Call EU Option with strike $k = 50$ at time T.

European Option : use in practice

European Options are widely used in finance, for mainly two reasons reasons :

- Hedging : prevent big losses in a portfolio, secure the price of a commodity[3] in advance, lock-in a currency rate in advance, ...
- Speculation

[3] E.g. Airline Companies use them to secure the price of oil

Value of European options

What is the *value* of an option contract ? It is the price people would agree to buy this contract at any time $t < T$.

- The price of an option at time t should reflect the expected payoff obtained at the maturity $\Phi(S(T))$

Several model exists to predict the price of assets, Black and Scholes used the Geometric Brownian Motion

Black-Scholes Model : Return on the underlying asset

- One of the most important component in the model is the prediction of the underlying price $S(t)$.
- Black and Scholes assumed that the dynamics of a stock follows a Geometric Brownian Motion :

$$\mathrm{d}S_t = \underbrace{S_t \mu(t)\mathrm{d}t}_{\text{drift, riskless part}} + \underbrace{S_t \sigma(t)\mathrm{d}W_t}_{\text{diffusion, risky part}} \tag{6}$$

Where W_t is a Wiener Process.

Simulations of Geometric Brownian Motion

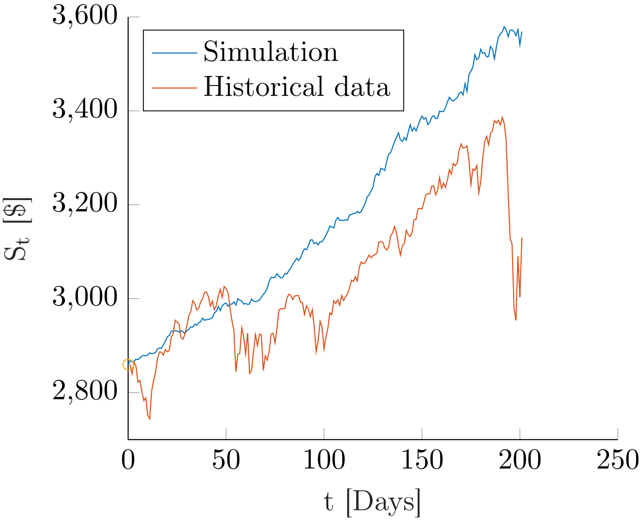

Figure 3: Simulation of a Geometric Brownian Motion with parameters estimaed from the S&P 500 over the last 200 days, average over 5000 trajectories.

Black-Scholes Model : Return on the underlying asset

- Why Geometric Brownian Motion ? Because *returns* (relative increments) on asset $\frac{\mathrm{d}S_t}{S_t}$ follow a drifted random walk, not the price itself.
- Assuming μ and σ constants the solution of this SDE reads:

$$S(t) = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t} \tag{7}$$

  (Blackboard derivation with Itô's Lemma.)
- Properties of this process :
  - Trajectories are continuous (it is a continuous time model),
  - The returns are independent,
  - The returns are identically distributed.

Value of an option

- The value of an option F can only depend on the current price of the underlying (as it is memoryless) and the current date : formally, we have $F = F(S_t, t)$. S_t being a stochastic process, one can apply Itô's lemma : (derivation on the blackboard)

$$\mathrm{d}F(S(t), t) = \left(\frac{\partial F}{\partial t} + \mu(t)S(t)\frac{\partial F}{\partial S} + \frac{1}{2}\sigma(t)^2 S(t)^2 \frac{\partial^2 F}{\partial S^2}\right)\mathrm{d}t + \sigma(t)S(t)\frac{\partial F}{\partial S}\mathrm{d}W(t)$$

- where the last term represent the stochastic (risky) component of the option's price.
- This is the **dynamics** of the option's price, but we need one more condition a ("reference point") to compute F : no arbitrage principle

Second equation : no arbitrage principle

- An arbitrage is a riskless strategy that has a positive payoff with probability 1.
- Such arbitrage are not allowed in the financial theory : this is the **no arbitrage principle**.
- It is an application of the Supply and Demand principle.

No arbitrage principle : riskless assets

- The no arbitrage principle implies that a riskless strategy should not pay more than lending money at the bank over the same period otherwise, it is underpriced.
- Bank account dynamics :

$$\mathrm{d}B(t) = rB(t)\mathrm{d}t \tag{8}$$

- Our goal is to construct a riskless portfolio which contains the option, in order to compare it with (8).

Construction of the portfolio

- It is possible to construct a riskless portfolio by buying one option, and selling[4] a certain amount Δ of the underlying.
- At time t, the value of this portfolio is

$$V(S_t, t) = \underbrace{F(S_t, t)}_{\text{Value of the option}} - \underbrace{\Delta \cdot S_t}_{\text{Value of stock}} \tag{9}$$

- The question is how to choose Δ to remove risk[5] ?

[4] In finance it is possible to hold both positive and "negative" quantity of asset. The first is said to be a *long* position and the second a *short* position.

[5] This way of defining the portfolio, called "Delta Hedging" is slightly different from the original derivation of Black and Scholes : Jean-Philippe Bouchaud and Marc Potters. *Theory of Financial Risk and Derivative Pricing: From Statistical Physics to Risk Management*. 2nd ed. Cambridge University Press, 2003

Hedged (riskless) portfolio

- Reminding the value of the portfolio :

$$V(S_t, t) = F(S_t, t) - \Delta \cdot S_t \tag{10}$$

- the goal is to protect the value of the portfolio[6] from the variation of S_t, hence we have to impose

$$0 = \frac{\partial V}{\partial S} = \frac{\partial F}{\partial S} - \Delta \implies \Delta = \frac{\partial F}{\partial S} \tag{11}$$

[6]We assume that the change in the portfolio value only comes from the variation of the asset's value and not from a change in portfolio's composition : it is said to be *self financing*.

Hedged portfolio : Verification with Itô's Lemma

- S_t being a stochastic process, one check that for $\Delta = \frac{\partial F}{\partial S}$ the dynamics of the portfolio is indeed riskless by applying Itô's Lemma :

$$\mathrm{d}V(s,t) = \mathrm{d}F(s,t) - \Delta \cdot \mathrm{d}S_t \tag{12}$$

$$= \left(\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 F}{\partial s^2} \right) \mathrm{d}t \tag{13}$$

(Blackboard derivation)

Hedged portfolio

- The dynamics of the portfolio is purely deterministic : it is indeed riskless for the choice $\Delta = \frac{\partial F}{\partial S}$
- Given the no arbitrage principle, this portfolio must have the same return than a bank account :

$$\mathrm{d}V(S,t) = rV(S,t)\mathrm{d}t \tag{14}$$

Summing up : our model

We now have the two components of our model :

1. The dynamics of the option's price

$$\mathrm{d}F(S,t) = \left(\frac{\partial F}{\partial t} + \mu S_t \frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 F}{\partial S^2}\right)\mathrm{d}t + \sigma S_t \frac{\partial F}{\partial S}\mathrm{d}W(t) \qquad (15)$$

2. The value of a riskless portfolio combining an option and the underlying :

$$\mathrm{d}V(s,t) = \mathrm{d}F(s,t) - \underbrace{\frac{\partial F}{\partial s}}_{\Delta}\mathrm{d}S(t) = rV(s,t)\mathrm{d}t \qquad (16)$$

Black-Scholes PDE

Combining the two previous equations gives the famous Black-Scholes PDE :

$$\frac{\partial F(s,t)}{\partial t} + rs\frac{\partial F(s,t)}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 F(s,t)}{\partial s^2} - rF(s,t) = 0, \quad (17)$$

$$F(S_T, T) = \Phi(S_T) \quad (18)$$

Black-Scholes PDE : Remarks

- The drift coefficient μ does not appear in the PDE. In fact the price F of the option is relative to the price of the underlying, hence its deterministic evolution does not play a big role. Only the volatility is important.
- If the interest rate of the bank is zero $r = 0$, the equation becomes :

$$\frac{\partial F}{\partial t} = -\frac{D}{2}\frac{\partial^2 F}{\partial s^2} \tag{19}$$

  which is a diffusion equation in "backward" time given the sign, with a *terminal condition* $\Phi(S_T)$ instead of initial condition, as we often have in physics.

Solving the equation

Remind the PDE :

$$\frac{\partial F(s,t)}{\partial t} + rs\frac{\partial F(s,t)}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 F(s,t)}{\partial s^2} - rF(s,t) = 0,$$
$$F(T,s) = \Phi(s)$$

- There are many ways of solving this equation
- One is to cast it into a backward Fokker-Planck equation for the transition probability $P(x,T|s,t)$:

$$\frac{\partial P(x,T|s,t)}{\partial t} + rs\frac{\partial P(x,T|s,t)}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 P(x,T|s,t)}{\partial s^2} = 0,$$
$$P(x,T|s,T) = \delta(x-s)$$

Solving the equation

In this case the solution reads :

$$F(s,t) = e^{-r(T-t)} \int \mathrm{d}x P(x,T|s,t) F(x,T) \qquad (20)$$

Where $F(x,T) \equiv \Phi(x)$ is the terminal value. For a call option, $\Phi(x) = (x-k)^+ = (x-k) \cdot \Theta(x-k)$

Solution to Black-Scholes PDE : call option

In the case of a call option $\Phi(S_T) = (S_T - k)^+$:

$$F(S_t, t) = S_t\, \phi(d_1) - ke^{-r(T-t)}\, \phi(d_2) \tag{21}$$

where

$$d_1 = \frac{\log\left(\frac{S_t}{k}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = d_1 - \sigma\sqrt{T-t}$$

$$\text{and } \phi(x) = \mathbb{P}(X \leq x) \text{ for } X \sim \mathcal{N}(0,1)$$

is the c.d.f of the normal law.

Black-Scholes Model assumptions

Black and Scholes derived a model to price European options, under several assumptions :

- The returns on the underlying asset are normally distributed.
- Stocks pay no dividend.
- There are no commissions and no transactions costs.
- The market is perfectly liquid : it is possible to buy or sell any amount of stock or options at any time, including fractional amount.

Solution to Black-Scholes PDE : put option

In the case of a call option $\Phi(S_T) = (S_T - k)^+$:

$$F(S_t, t) = -S_t\, \phi(-d_1) - ke^{-r(T-t)}\, \phi(-d_2) \tag{22}$$

where

$$d_1 = \frac{\log\left(\frac{S_t}{k}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = d_1 - \sigma\sqrt{T-t}$$

$$\text{and } \phi(x) = \mathbb{P}(X \leq x) \text{ for } X \sim \mathcal{N}(0,1)$$

is the c.d.f of the normal law.

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